(#) Odrediti sve vrijednosti parametra m tako da vektori $\vec{a}=\begin{pmatrix}m-2\\1\\2\end{pmatrix}$, $\vec{b}=\begin{pmatrix}m-2\\m-2\\3\end{pmatrix}$, $\vec{c}=\begin{pmatrix}m-2\\1\\m-2\end{pmatrix}$ nisu baza (ne čine bazu) vektorskog prostora $\mathbb{R}^3$. Za najveću dobijenu vrijednost parametra m izraziti vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rj.-uputa.

Vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ neće činiti bazu vektorskog prostora $\mathbb{R}^3$ ako su linearno zavisni, a oni su linearno zavisni ako postoje brojevi α, β i γ (ne svi nula) taki da $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=0$,

$$\alpha\begin{pmatrix}m-2\\1\\2\end{pmatrix}+\beta\begin{pmatrix}m-2\\m-2\\3\end{pmatrix}+\gamma\begin{pmatrix}m-2\\1\\m-2\end{pmatrix}=0 \iff \underbrace{\begin{pmatrix}m-2 & m-2 & m-2\\1 & m-2 & 1\\2 & 3 & m-2\end{pmatrix}}_{=M}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

a ovaj sistem će imati netrivijalna rješenja za

$\det M \neq 0$

$$\det M=\begin{vmatrix}m-2 & m-2 & m-2\\1 & m-2 & 1\\2 & 3 & m-2\end{vmatrix}=(m-2)\begin{vmatrix}1 & 1 & 1\\1 & m-2 & 1\\2 & 3 & m-2\end{vmatrix}=\ldots=(m-2)(m-3)(m-4)$$

Za $m\in\{2,3,4\}$ dati vektori nisu baza prostora $\mathbb{R}^3$.

Za $m=4$ imamo $\vec{a}=\begin{pmatrix}2\\1\\2\end{pmatrix}$, $\vec{b}=\begin{pmatrix}2\\2\\3\end{pmatrix}$, $\vec{c}=\begin{pmatrix}2\\1\\2\end{pmatrix}$

$$\vec{c}=\eta\begin{pmatrix}2\\1\\2\end{pmatrix}+\mu\begin{pmatrix}2\\2\\3\end{pmatrix} \Rightarrow \begin{matrix}\eta=1\\ \mu=0\end{matrix} \qquad \vec{c}=\vec{a}+0\cdot\vec{b}.$$

(#) Ako je $\{\vec{a_1},\vec{a_2},\vec{a_3}\}$ jedna baza vektorskog prostora $\mathbb{R}^3$, dokazati da i vektori $\vec{b_1}=\vec{a_2}+3\vec{a_3}$, $\vec{b_2}=\vec{a_1}+\vec{a_2}+2\vec{a_3}$ i $\vec{b_3}=2\vec{a_1}+2\vec{a_2}+6\vec{a_3}$ takoder čine baza prostora $\mathbb{R}^3$ i izraziti vektor $\vec{c}=-\vec{a_1}+\vec{a_2}+2\vec{a_3}$ preko vektora baze $\{\vec{b_1},\vec{b_2},\vec{b_3}\}$.

Rj.-upute:

Vektori $\vec{b_1},\vec{b_2}$ i $\vec{b_3}$ će činiti bazu prostora $\mathbb{R}^3$ ako su linearno nezavisni tj. ako je jedino rješenje sistema

$$\alpha\vec{b_1}+\beta\vec{b_2}+\gamma\vec{b_3}=\vec{0}$$

trivijalno rješenje $\alpha=\beta=\gamma=0$. Posmatrajmo dati sistem

$$\alpha(\vec{a_2}+3\vec{a_3})+\beta(\vec{a_1}+\vec{a_2}+2\vec{a_3})+\gamma(2\vec{a_1}+2\vec{a_2}+6\vec{a_3})=\vec{0}$$

$$(0+\beta+2\gamma)\vec{a_1}+(\alpha+\beta+2\gamma)\vec{a_2}+(3\alpha+2\beta+6\gamma)\vec{a_3}=\vec{0}$$

$$\begin{aligned}\beta+2\gamma&=0\\ \alpha+\beta+2\gamma&=0\\ 3\alpha+2\beta+6\gamma&=0\end{aligned} \iff \underbrace{\begin{pmatrix}0&1&2\\1&1&2\\3&2&6\end{pmatrix}}_{=M}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\det M=\begin{vmatrix}0&1&2\\1&1&2\\3&2&6\end{vmatrix}=\ldots=-2\neq 0 \Rightarrow$$ vektori $\vec{b_1},\vec{b_2}$ i $\vec{b_3}$ su linearno nezavisni i oni čine bazu prostora $\mathbb{R}^3$

Odredimo još konstante c_1, c_2 i c_3 t.d. $\vec{c}=c_1\vec{b_1}+c_2\vec{b_2}+c_3\vec{b_3}$

$$-\vec{a_1}+\vec{a_2}+2\vec{a_3}=c_1(a_2+3\vec{a_3})+c_2(\vec{a_1}+\vec{a_2}+2\vec{a_3})+c_3(2\vec{a_1}+2\vec{a_2}+6\vec{a_3})$$

$$\begin{aligned}c_2+2c_3&=-1\\ c_1+c_2+2c_3&=1\\ 3c_1+2c_2+6c_3&=2\end{aligned} \iff \begin{bmatrix}0&1&2\\1&1&2\\3&2&6\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-1\\1\\2\end{bmatrix} \Rightarrow \begin{aligned}c_1&=2\\c_2&=1\\c_3&=-1\end{aligned}$$

(#) Odrediti sve vrijednosti parametra m tako da vektori $\vec{a}=\begin{pmatrix}m-1\\m-1\\m-1\end{pmatrix}$, $\vec{b}=\begin{pmatrix}1\\m-1\\1\end{pmatrix}$, $\vec{c}=\begin{pmatrix}2\\3\\m-1\end{pmatrix}$ nisu baza (ne čine bazu) vektorskog prostora $\mathbb{R}^3$. Za najveću dobijenu vrijednost parametra m izraziti vektor $\vec{c}$ kao linearnu kombinaciju vektora $\vec{a}$ i $\vec{b}$.

Rj.-uputa:

Za vektore $\vec{a}$, $\vec{b}$ i $\vec{c}$ kažemo da su linearno zavisni ako postoje konstante α, β i γ (ne sve jednake nuli) t.d. $\alpha\vec{a}\cancel{\beta\vec{b}}+\gamma\vec{c}=\vec{0}$

$$\alpha\begin{pmatrix}m-1\\m-1\\m-1\end{pmatrix}+\beta\begin{pmatrix}1\\m-1\\1\end{pmatrix}+\gamma\begin{pmatrix}2\\3\\m-1\end{pmatrix}=0 \iff \underbrace{\begin{bmatrix}m-1 & 1 & 2\\ m-1 & m-1 & 3\\ m+1 & 1 & m-1\end{bmatrix}}_{M}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Sistem će imati jedinstveno rješenje kada je $\det M=0$ tj. za $\det M\neq 0$ vektori su linearno zavisni i neće formirati bazu za $\mathbb{R}^3$.

$$\det M=\begin{vmatrix}m-1 & 1 & 2\\ m-1 & m-1 & 3\\ m-1 & 1 & m-1\end{vmatrix}=(m-1)\begin{vmatrix}1 & 1 & 2\\ 1 & m-1 & 3\\ 1 & 1 & m-1\end{vmatrix}=\ldots=(m-1)(m-2)(m-3)$$

Za $m\in\{1,2,3\}$ dati vektori nisu baza prostora $\mathbb{R}^3$.

Za $m=3$ imamo $\vec{a}=\begin{pmatrix}2\\2\\2\end{pmatrix}$, $\vec{b}=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\vec{c}=\begin{pmatrix}2\\3\\2\end{pmatrix}$

$$\vec{c}=\eta\begin{pmatrix}2\\2\\2\end{pmatrix}+\mu\begin{pmatrix}1\\2\\1\end{pmatrix} \Rightarrow \begin{matrix}\eta=\frac{1}{2}\\ \mu=1\end{matrix} \quad \text{tj. } \vec{c}=\frac{1}{2}\vec{a}+\vec{b}$$

(#) Ako je $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ jedna baza vektorskog prostora $\mathbb{R}^3$, dokazati da i vektori $\vec{b_1} = \vec{a_1} + 2\vec{a_2} + 3\vec{a_3}$, $\vec{b_2} = \vec{a_1} + \vec{a_2} + 2\vec{a_3}$ i $\vec{b_3} = 2\vec{a_1} + \vec{a_2} + 4\vec{a_3}$ također čine bazu prostora $\mathbb{R}^3$ i izraziti vektor $\vec{c} = \vec{a_1} + \vec{a_2} + \vec{a_3}$ preko vektora baze $\{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$.

Rješenje-upute:

Vektori $\vec{b_1}, \vec{b_2}$ i $\vec{b_3}$ će činiti bazu prostora $\mathbb{R}^3$ ako su linearno nezavisni. Drugim riječima ako je jedino rješenje sistema $\alpha\vec{b_1} + \beta\vec{b_2} + \gamma\vec{b_3} = \vec{0}$

trivijalno rješenje $\alpha = \beta = \gamma = 0$. Pa posmatrajmo dati sistem

$$\alpha\underbrace{(\vec{a_1} + 2\vec{a_2} + 3\vec{a_3})}_{=\vec{b_1}} + \beta\underbrace{(\vec{a_1} + \vec{a_2} + 2\vec{a_3})}_{=\vec{b_2}} + \gamma\underbrace{(2\vec{a_1} + \vec{a_2} + 4\vec{a_3})}_{=\vec{b_3}} = \vec{0}$$

$$(\alpha + \beta + 2\gamma)\vec{a_1} + (2\alpha + \beta + \gamma)\vec{a_2} + (3\alpha + 2\beta + 4\gamma)\vec{a_3} = \vec{0}$$

$$\begin{aligned} \alpha + \beta + 2\gamma &= 0 \\ 2\alpha + \beta + \gamma &= 0 \\ 3\alpha + 2\beta + 4\gamma &= 0 \end{aligned} \quad \Leftrightarrow \quad \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}}_{=M} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det M = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = \ldots = -1 \neq 0 \Rightarrow$$ vektori $\vec{b_1}, \vec{b_2}$ i $\vec{b_3}$ su linearno nezavisni

Odredimo konstante c_1, c_2 i c_3 t.d. $\vec{c} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3}$

$$\vec{a_1} + \vec{a_2} + \vec{a_3} = c_1(\vec{a_1} + 2\vec{a_2} + 3\vec{a_3}) + c_2(\vec{a_1} + \vec{a_2} + 2\vec{a_3}) + c_3(2\vec{a_1} + \vec{a_2} + 4\vec{a_3})$$

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 1 \\ 2c_1 + c_2 + c_3 &= 1 \\ 3c_1 + 2c_2 + 4c_3 &= 1 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow c_1 = -1,\ c_2 = 4,\ c_3 = -1$$

Ispitati f-ju i nacrtati njen grafik: $y=\frac{\ln^2 x+1}{x^2}$.

R: definiciono područje

$x\neq 0$ i $x>0$

$\mathcal{D}: x\in(0,+\infty)$

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično

$\Rightarrow$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ akko $\ln^2 x+1=0$

$(\ln x)^2=-1$

f-ja nema nulu

$f(0)$ nije definisano

f-ja ne siječe y-osu

$\ln^2 x+1>0 \quad \forall x\in\mathcal{D}$

$x^2>0 \quad \forall x\in\mathcal{D}$

f-ja je uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x\leq 0$ f-ja nije definisana

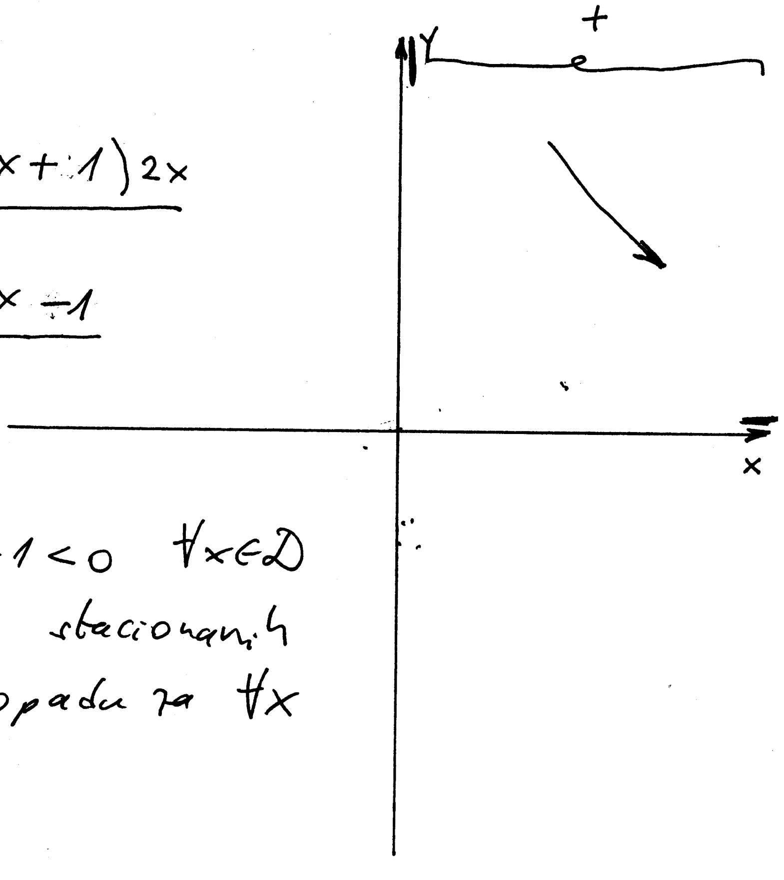
$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{\ln^2 x+1}{x^2}=\frac{+\infty}{0+}=+\infty \quad\Rightarrow\quad x=0 \text{ je vertikalna asimptota}$$

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{\ln^2 x+1}{x^2}\left(=\frac{\infty}{\infty}\right)\overset{L.P.}{=}\lim_{x\to+\infty}\frac{2\ln x\cdot\frac{1}{x}}{2x}=\lim_{x\to+\infty}\frac{\ln x}{x^2}\left(\frac{\infty}{\infty}\right)\overset{L.P.}{=}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}}{2x}=0 \quad\Rightarrow\quad y=0 \text{ je horizontalna asimptota}$$

f-ja nema kosu asimptotu

počinjemo skicirati grafik



rast i opadanje

$$y'=\left(\frac{\ln^2 x+1}{x^2}\right)'=\frac{2\ln x\cdot\frac{1}{x}\cdot x^2-(\ln^2 x+1)2x}{x^4}$$

$$=\frac{2x(\ln x-\ln^2 x-1)}{x^4}=2\,\frac{\ln x-\ln^2 x-1}{x^3}$$

$y'=0$ akko $-\ln^2 x+\ln x-1=0$

$\ln x=t$

$-t^2+t-1=0$

$t^2-t+1=0$

$D=1-4<0$

$-\ln^2 x+\ln x-1<0 \quad \forall x\in\mathcal{D}$

f-ja nema stacionarnih tački i opada za $\forall x$

ekstremi f-je

f-ja nema stacionarnih tački ⇒ f-ja nema ekstrema

prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = 2\left(\frac{\ln x - \ln^2 x - 1}{x^3}\right)' = 2\,\frac{\left(\frac{1}{x} - 2\ln x\cdot\frac{1}{x}\right)\cancel{x^3} - (\ln x - \ln^2 x - 1)\cdot 3\cancel{x^2}}{x^{\cancel{6}4}} =$$

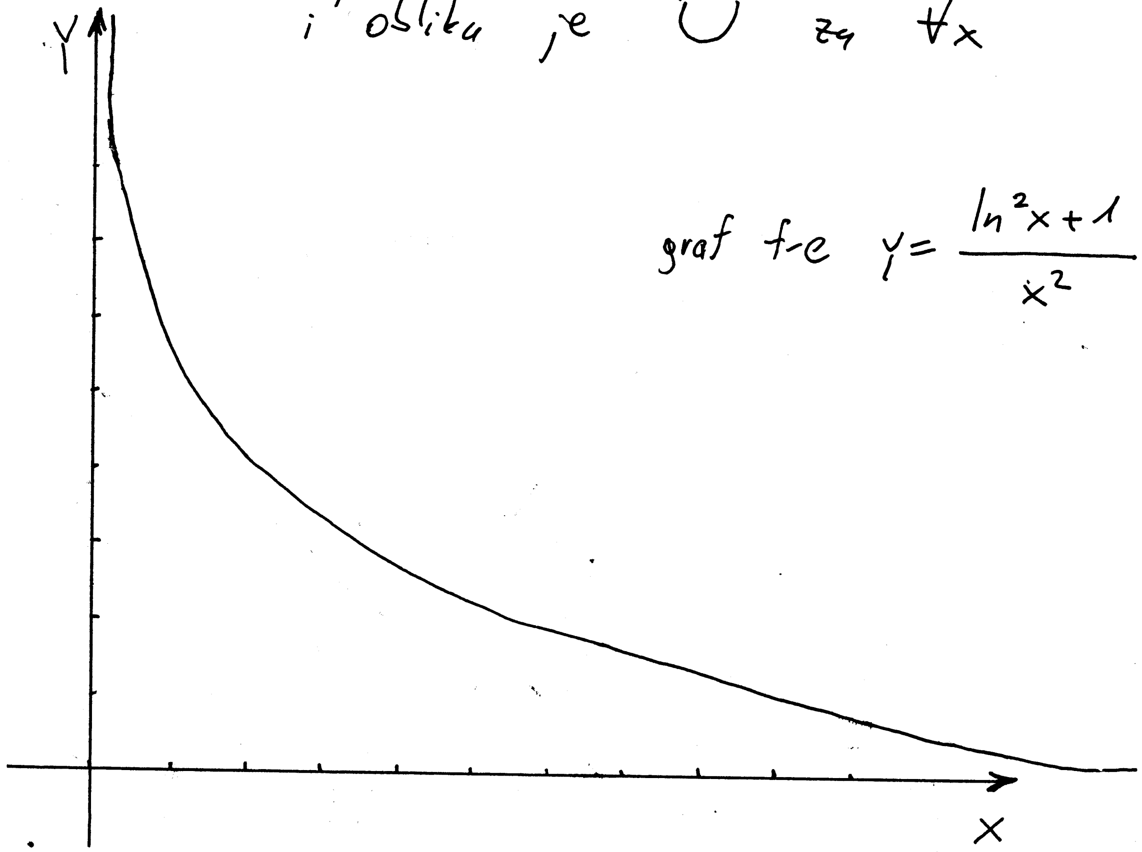
$$= 2\,\frac{1\boxed{-2\ln x}\boxed{-3\ln x}+\underline{\underline{3\ln^2 x}}+3}{x^4} = 2\,\frac{3\ln^2 x - 5\ln x + 4}{x^4}.$$

$3\ln^2 x - 5\ln x + 4 = 0$

$\ln x = t \qquad 3t^2 - 5t + 4 = 0 \quad \Rightarrow \quad 3\ln^2 x - 5\ln x + 4 > 0 \quad \forall x$

$D = 25 - 48 < 0 \qquad\qquad x^4 > 0 \quad \forall x$

$y'' > 0 \quad \forall x \in D \quad \Rightarrow$ f-ja nema prevojnih tački i oblika je $\cup$ za $\forall x$



graf f-je $y = \dfrac{\ln^2 x + 1}{x^2}$

(#) Ispitati f-ju i nacrtati njen grafik: $y=\frac{x^3-2}{2x^2}$.

Rj. definiciono područje
$\mathcal{D}: x\neq 0$

parnost (neparnost), periodičnost

$$f(-x)=\frac{(-x)^3-2}{2(-x)^2}=\frac{-x^3-2}{2x^2}\neq \pm f(x)$$

· f-ja nije ni parna ni neparna
f-ja nije periodična

nule, presjek sa y-osom, znak

$y=0$ akko $x^3-2=0$

$x=\sqrt[3]{2}\approx 1,26$

$(\sqrt[3]{2},0)$ je nula f-je

$f(0)=$ nije definisano
f-ja ne siječe y-osu

$2x^2>0\ \forall x\in\mathcal{D}$

$$\left.\begin{array}{lll} y>0 & \text{za} & x>\sqrt[3]{2} \\ y<0 & \text{za} & x<\sqrt[3]{2} \end{array}\right\}\ \text{znak f-je.}$$

ponašanje na krajevima intervala definisanosti i asimptote

· za $x=0$ f-ja ima prekid

$$\left.\begin{array}{l} \lim\limits_{x\to0-} f(x)=\lim\limits_{x\to0-}\frac{x^3-2}{2x^2}=\frac{(0-)^3-2}{2(0-)^2}=\frac{-2-0}{0+}=-\infty \\ \lim\limits_{x\to0+} f(x)=\lim\limits_{x\to0+}\frac{(0+)^3-2}{2(0+)^2}=\frac{-2+0}{+0}=-\infty \end{array}\right\}\Rightarrow \begin{array}{c} x=0 \\ \text{je V. A.} \end{array}$$

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^3-2\ /:x^2}{2x^2\ /:x^2}=\pm\infty$$ f-ja nema H. A.

Tražimo kosu asimptotu u obliku $y=kx+n$.

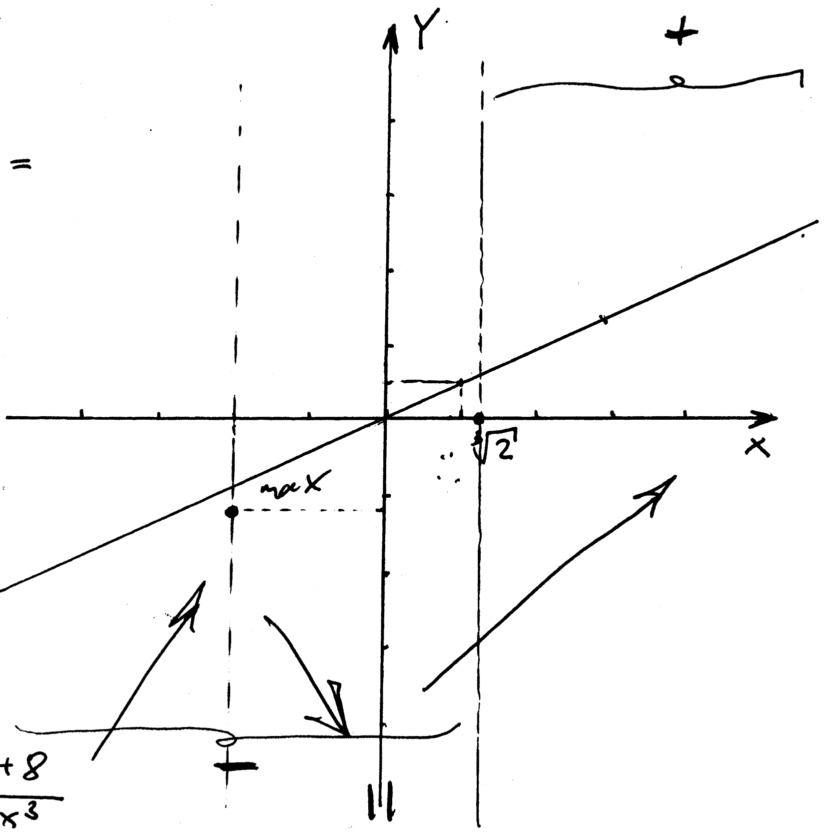
$$k=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^3-2\ /:x^3}{2x^3\ /:x^3}=\frac{1}{2}$$

$$n=\lim_{x\to\infty}\left[f(x)-kx\right]=\lim_{x\to\infty}\left[\frac{x^3-2}{2x^2}-\frac{1}{2}x\right]=$$

$$=\lim_{x\to\infty}\frac{x^3-2-x^3}{2x^2}=\lim_{x\to\infty}\frac{-2}{2x^2}=0$$

kosa asimptota je $y=\frac{1}{2}x$

Poslije ovog koraka počinjemo skicirati grafik.



rast i opadanje

$$y'=\left(\frac{x^3-2}{2x^2}\right)'=\frac{3x^2\cdot 2x^2-(x^3-2)4x}{4x^4}=$$

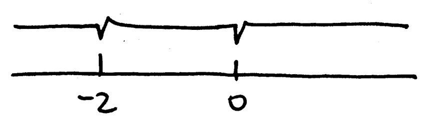
$$=\frac{6x^4-4x^4+8x}{4x^4}=\frac{2x^4+8x}{4x^4}=\frac{x^3+8}{2x^3}$$

$y' = \frac{x^3+8}{2x^3}$, $y'=0$ akko $x^3+8=0$

$x^3=-8$

$x=-2$

prekidi y + nule y'



x	$(-\infty,-2)$	$(-2,0)$	$(0,+\infty)$
y'	+	● −	+
y	↗	↘	↗
		max	N.D.

$$f(-2) = \frac{(-2)^3-2}{2(-2)^2} = \frac{-10}{8} = -\frac{5}{4} \approx -1{,}25$$

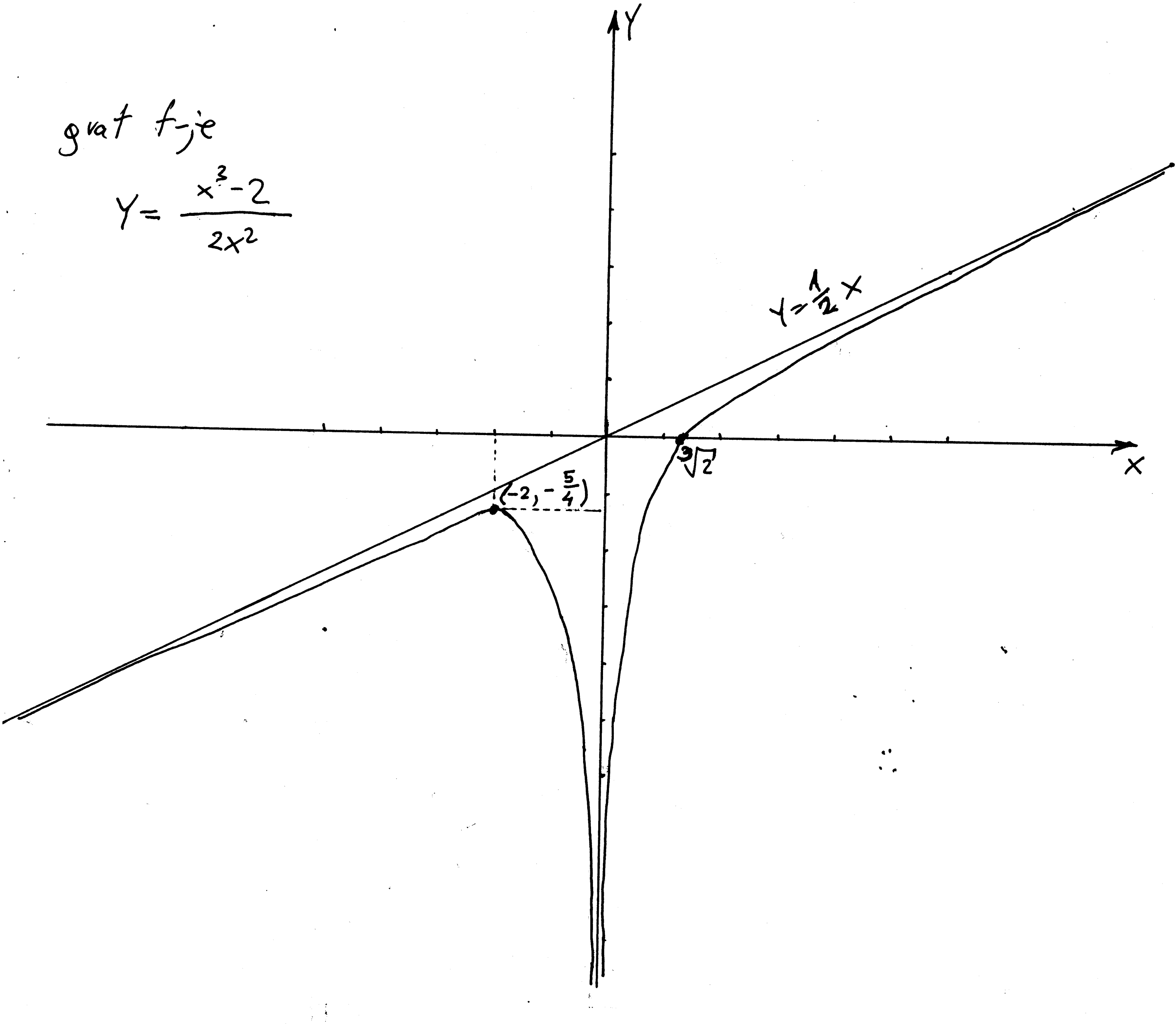
prevojne tačke i intervali konveksnosti i konkavnosti

$$y'' = \left(\frac{x^3+8}{2x^3}\right)' = \frac{3x^2\cdot 2x^3-(x^3+8)\cdot 6x^2}{4x^6} = \frac{6x^5-6x^5-48}{4x^6} = \frac{-48}{4x^6} = -\frac{12}{x^6} < 0$$

F-ja nema prevojnih tački i uvijek je negativna što znači uvijek je $\cap$ oblika.

graf f-je

$$y = \frac{x^3-2}{2x^2}$$



(#) Ispitati f-ju i nacrtati joj grafik $y = \ln \frac{x^2-3x+2}{x^2+1}$.

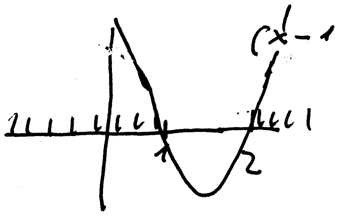
Rj: definiciono područje

Kako je $x^2+1>0 \ \forall x \in R$

to iz $\frac{x^2-3x+2}{x^2+1} > 0 \Rightarrow$

treba da bude $x^2-3x+2>0$

$(x-1)(x-2)>0$



$\mathcal{D}: x \in (-\infty, 1) \cup (2, +\infty)$

nule, presjek sa y-osom, znak

$y=0 \Rightarrow \ln \frac{x^2-3x+2}{x^2+1} = 0$

$\Rightarrow \frac{x^2-3x+2}{x^2+1} = 1 \quad / \cdot x^2+1$

$\cancel{x^2}-3x+2 = \cancel{x^2}+1$

$3x = 1 \Rightarrow x = \frac{1}{3}$

$(\frac{1}{3}, 0)$ je nula f-je

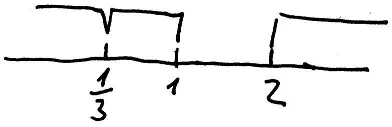
$y(0) = \ln 2 \approx 0,6931$

$(0, \ln 2)$ je presjek sa y-osom

parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično ⇒ f-ja nije ni parna ni neparna

f-ja nije periodična



← prekidi y + nule y

x	$(-\infty, \frac{1}{3})$	$(\frac{1}{3}, 1)$	$(2, +\infty)$
y	+	−	−

znak f-je

ponašanje na krajevima intervala definisanosti i asimptote

f-ja ima prekid za $x=1$ i $x=2$

$\lim_{x \to 1-0} f(x) = \lim_{x \to 1-0} \ln \frac{x^2-3x+2}{x^2+1} = \ln \frac{(1-0)^2-3(1-0)+2}{(1-0)^2+1} = \ln(0_+) = -\infty \Rightarrow$

$\Rightarrow x=1$ je V. A. (sa lijeve str.)

$\lim_{x \to 2+0} f(x) = \lim_{x \to 2+0} \ln \frac{x^2-3x+2}{x^2+1} = \ln(0+) = -\infty$

$\Rightarrow x=2$ je V. A. (sa desne strane)

$\lim_{x \to \mp\infty} f(x) = \lim_{x \to \mp\infty} \ln \frac{x^2-3x+2}{x^2+1} = \lim_{x \to \mp\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{1}{x^2}} = \ln 1 = 0$

$\Rightarrow y=0$ je H. A.

K. A. nema

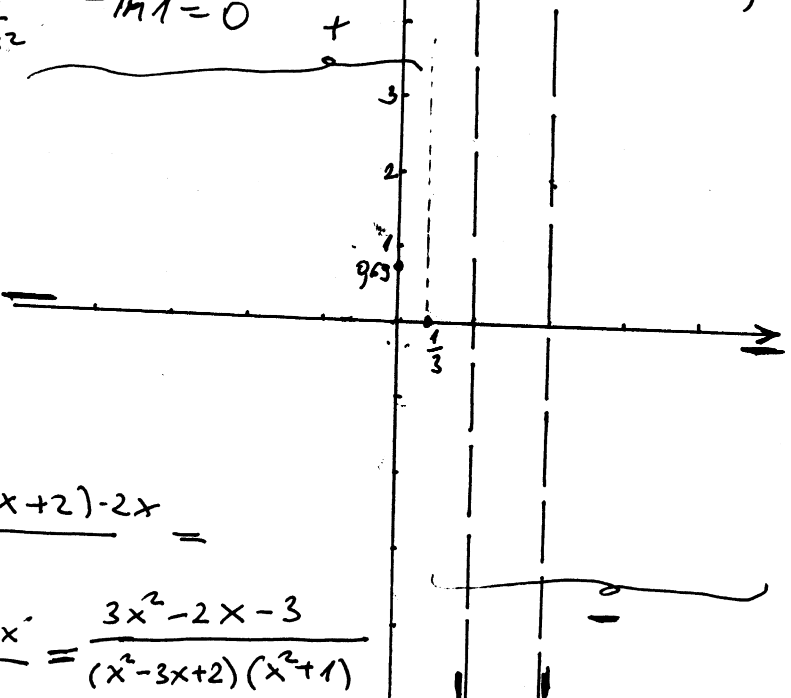
počinjemo sa skiciranjem grafa

rast i opadanje

$y' = \frac{x^2+1}{x^2-3x+2} \cdot \left(\frac{x^2-3x+2}{x^2+1} \right)'$

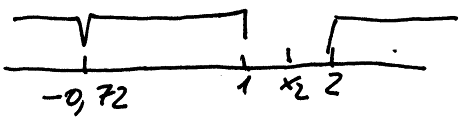
$y' = \frac{\cancel{x^2+1}}{x^2-3x+2} \cdot \frac{(2x-3)(x^2+1)-(x^2-3x+2) \cdot 2x}{(x^2+1)^{\cancel{2}}} =$

$= \frac{\cancel{2x^3}+2x-3x^2-3-\cancel{2x^3}+6x^2-4x}{(x^2-3x+2)(x^2+1)} = \frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}$



$y'=0 \Rightarrow 3x^2-2x-3=0 \Rightarrow x_{1,2}=\frac{2\pm\sqrt{4+36}}{6}$

$x_{1,2}=\frac{2\pm2\sqrt{10}}{6}=\frac{1\pm\sqrt{10}}{3}$



$x_1=\frac{1+\sqrt{10}}{3}\approx 1,387\notin\mathcal{D}$

$x_2=\frac{1-\sqrt{10}}{3}\approx -0,721\in\mathcal{D}$

x	$\left(-\infty, \frac{1-\sqrt{10}}{3}\right)$	$\left(\frac{1-\sqrt{10}}{3}, 1\right)$	$(2, +\infty)$
y'	+	−	+
y	↗	↘	↗

max

ekstremi f-je

$f\left(\frac{1-\sqrt{10}}{3}\right)\approx 1,016$

F-ja ima maksimum u tački $(-0,72; 1,02)$

prevojne tačke i intervali konveksnosti i konkavnosti

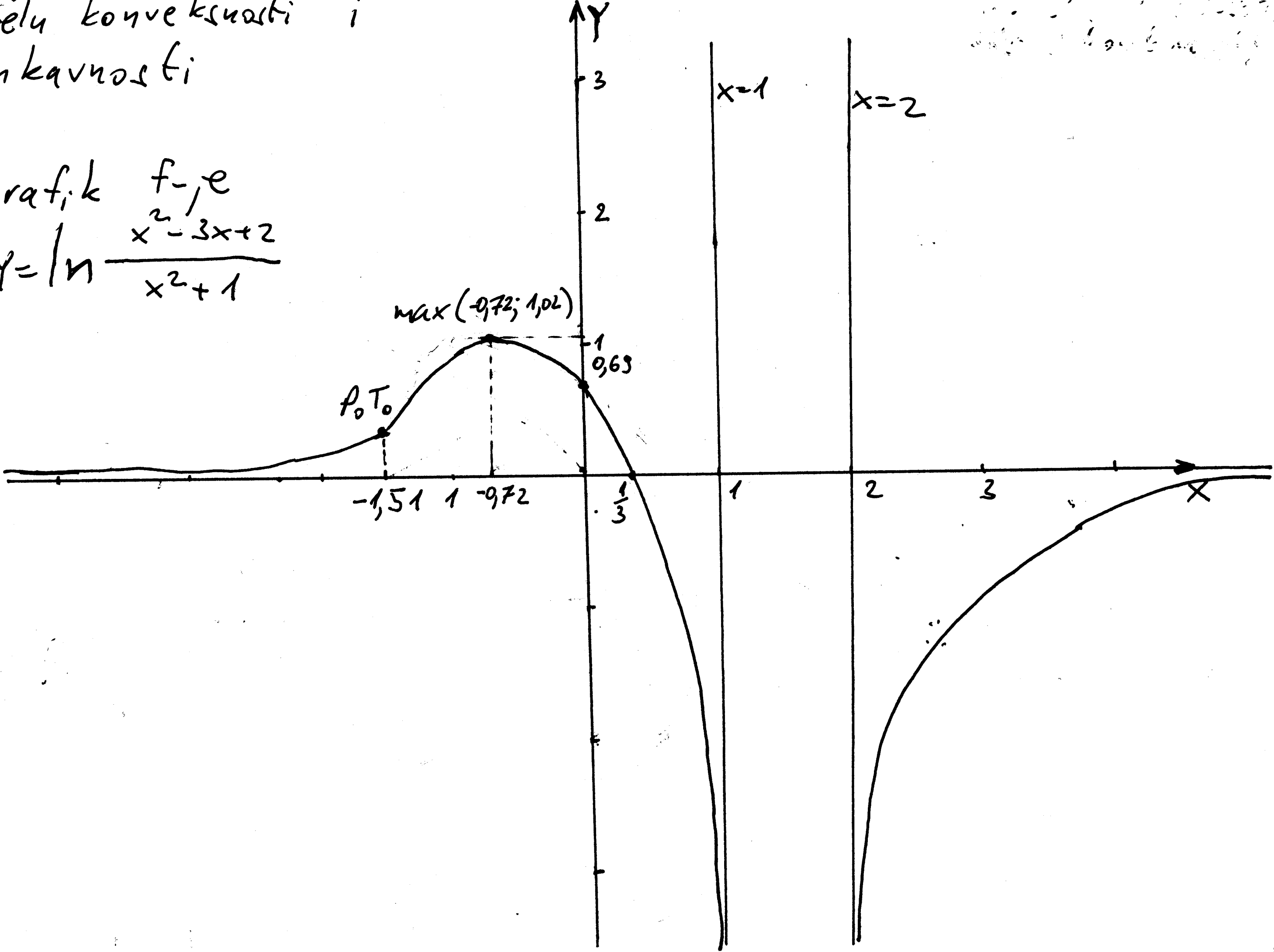
$y''=\left(\frac{3x^2-2x-3}{(x^2-3x+2)(x^2+1)}\right)'=$ ZA VJEŽBU ... $=\frac{-6x^5+15x^4-30x^2+30x-13}{(x^2-3x+2)^2(x^2+1)^2}$

$y''=0$ akko $x=-1,5166$ (izračunato uz pomoć kalkulatora)

Kako je brojnik u y'' previše složen nije potrebno praviti tabelu konveksnosti i konkavnosti

grafik f-je

$y=\ln\frac{x^2-3x+2}{x^2+1}$



(#) Ispitati f-ju i nacrtati njen grafik $Y = \frac{x^2+10}{x^2+4x+4}$.

R: $Y = \frac{x^2+10}{x^2+4x+4} = \frac{x^2+10}{(x+2)^2}$

definiciono područje

$x+2 \neq 0$ $\quad \mathcal{D}: x \in (-\infty, -2) \cup (-2, +\infty)$

$x \neq -2$

·parnost (neparnost), periodičnost

$\mathcal{D}$ nije simetrično ⇒ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa Y-osom i znak f-je

$Y=0 \Rightarrow x^2+10=0$

Kako je $x^2+10>0 \ \forall x \in \mathcal{D}$ to f-ja nema nule

$f(0) = \frac{0+10}{(0+2)^2} = \frac{10}{4} = \frac{5}{2}$

$(0, \frac{5}{2})$ je presjek sa Y-osom

$x^2+10>0 \ \forall x \in \mathcal{D}$ $\quad$ f-ja je

$(x+2)^2>0 \ \forall x \in \mathcal{D}$ $\quad$ uvijek pozitivna

ponašanje na krajevima intervala definisanosti i asimptote

za $x=-2$ f-ja ima prekid

$\lim\limits_{x \to -2-0} f(x) = \lim\limits_{x \to -2-0} \frac{x^2+10}{(x+2)^2} = \frac{(-2-0)^2+10}{(-2-0+2)^2} = \frac{14+0}{+0} = +\infty$ $\Rightarrow$ $x=-2$ je V. A. (sa lijeve strane)

$\lim\limits_{x \to -2+0} f(x) = \lim\limits_{x \to -2+0} \frac{x^2+10}{(x+2)^2} = \frac{(-2+0)^2+10}{(-2+0+2)^2} = \frac{14-0}{+0} = +\infty$ $\rightarrow$ $x=-2$ je V. A. (sa desne strane)

$\lim\limits_{x \to \mp\infty} f(x) = \lim\limits_{x \to \mp\infty} \frac{x^2+10 :x^2}{x^2+4x+4 :x^2} = \lim\limits_{x \to \mp\infty} \frac{1+\overset{\nearrow 0}{\frac{10}{x^2}}}{1+\underset{\swarrow 0}{\frac{4}{x}}+\underset{\swarrow 0}{\frac{4}{x^2}}} = 1$ $\Rightarrow$ $Y=1$ je H. A.

f-ja nema kosu asimptotu

Poslije ovog koraka počinjemo skicirati grafik.

rast i opadanje

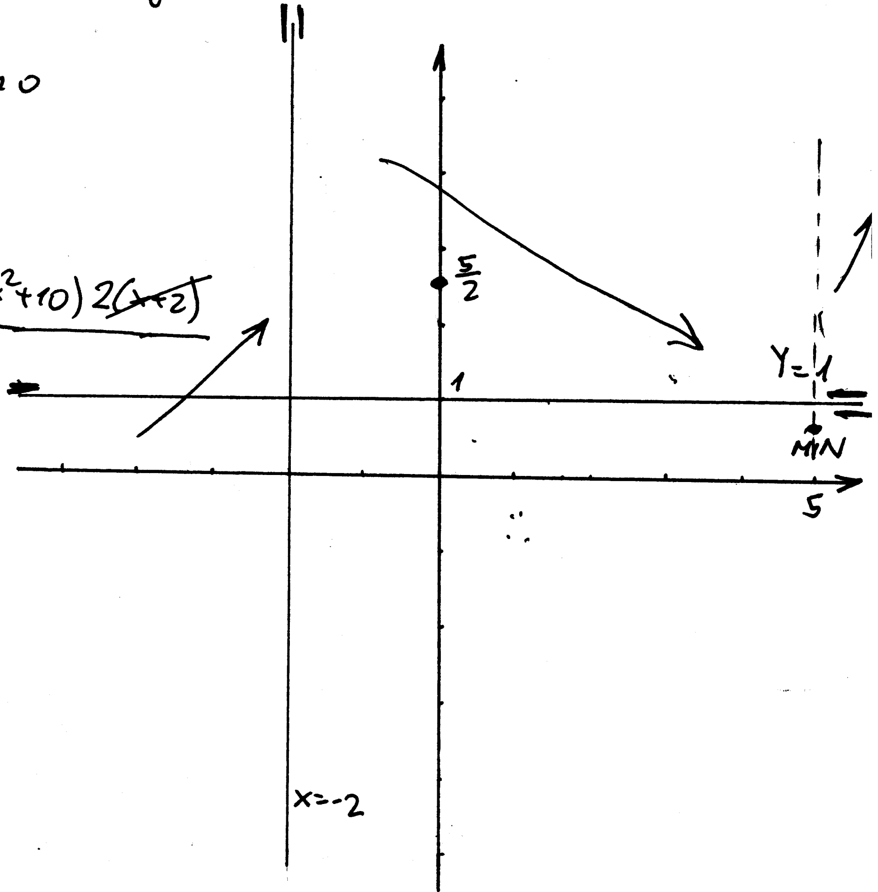
$Y' = \left(\frac{x^2+10}{(x+2)^2}\right)' = \frac{2x \cdot (x+2)^{\cancel{2}} - (x^2+10)2\cancel{(x+2)}}{(x+2)^{\cancel{4}\,3}}$

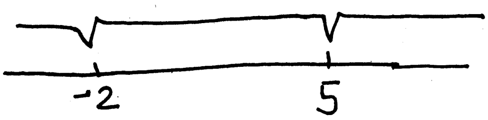
$Y' = \frac{\cancel{2x^2}+4x-\cancel{2x^2}-20}{(x+2)^3}$

$Y' = \frac{4x-20}{(x+2)^3} = 4\frac{x-5}{(x+2)^3}$

$Y'=0$ akko $x-5=0$

$x=5$





← prekidi Y + nule Y'

x	$(-\infty,-2)$	$(-2,5)$	$(5,+\infty)$
Y'	+	−	+
Y	↗	↘	↗

min

rast i opadanje

ekstremi f-je

Stacionarna tačka je $x=5$.
Na osnovu tabele rasta i opadanja vidimo da f-ja u toj tački ima ekstrem i to minimum

$f(5)=\frac{25+10}{7^2}=\frac{35}{49}\approx 0{,}71$ $\left(5,\frac{35}{49}\right)$ je tačka minimuma

prevojne tačke i intervali konveksnosti i konkavnosti

$$Y''=\left(4\frac{x-5}{(x+2)^3}\right)'=4\frac{1\cdot(x+2)^{\cancel{3}}-(x-5)3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}4}}=4\frac{x+2-3x+15}{(x+2)^4}.$$

$$Y''=4\frac{-2x+17}{(x+2)^4}=-4\frac{2x-17}{(x+2)^4}$$

← nule Y'' + prekidi Y

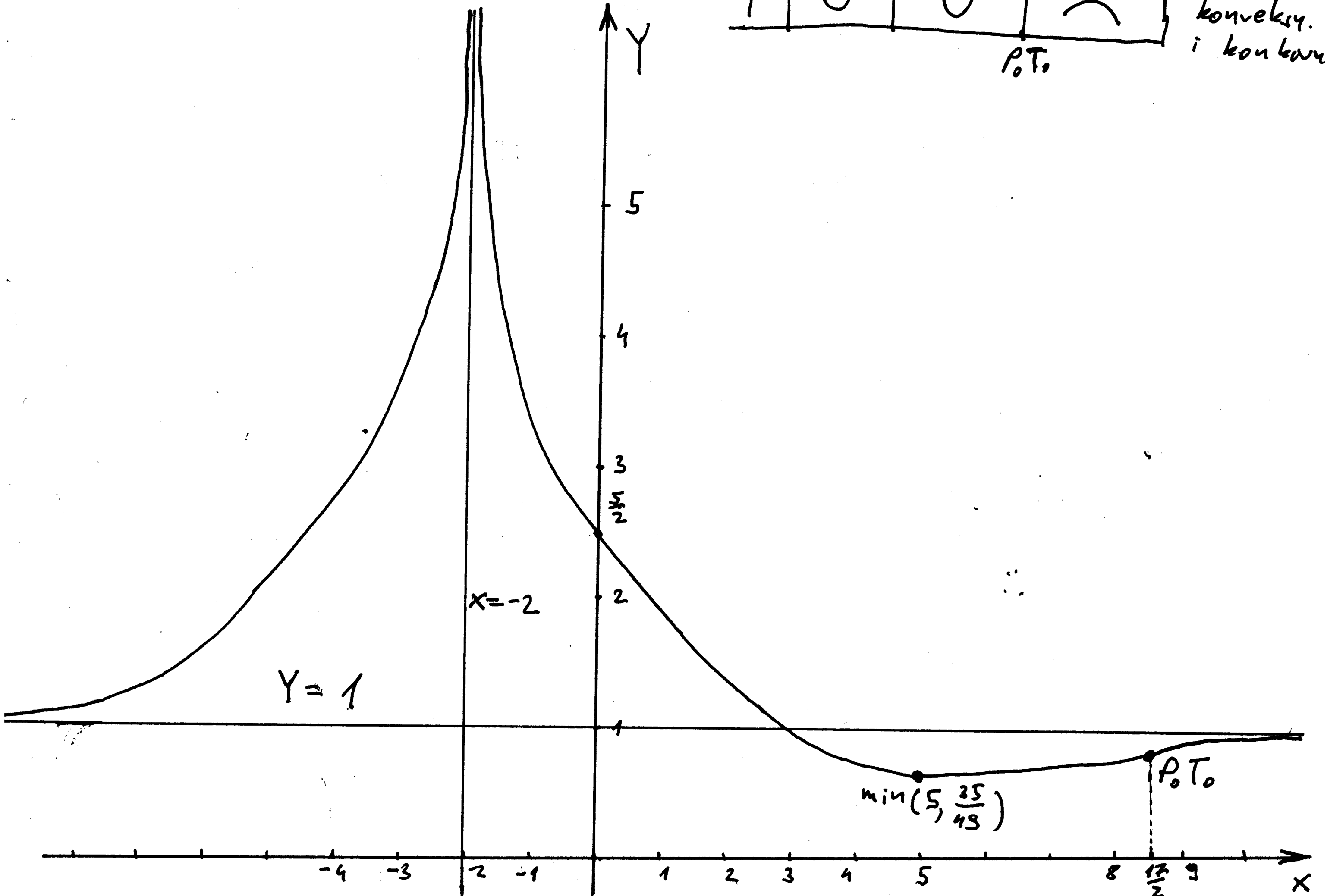
$Y''=0$ akko $2x-17=0$

$x=\frac{17}{2}$

x	$(-\infty,-2)$	$\left(-2,\frac{17}{2}\right)$	$\left(\frac{17}{2},+\infty\right)$
Y''	+	+	−
Y	∪	∪	∩

P_0T_0

intervali konveksn. i konkavn.



(#) Odrediti površinu figure ograničene hiperbolom $xy=4$ i pravom $y=-x-5$.

Rj.: -upute

Odredimo presječne tačke datih krivih i skicirajmo sliku.

$$\begin{aligned} xy&=4\\ y&=-x-5 \end{aligned}$$

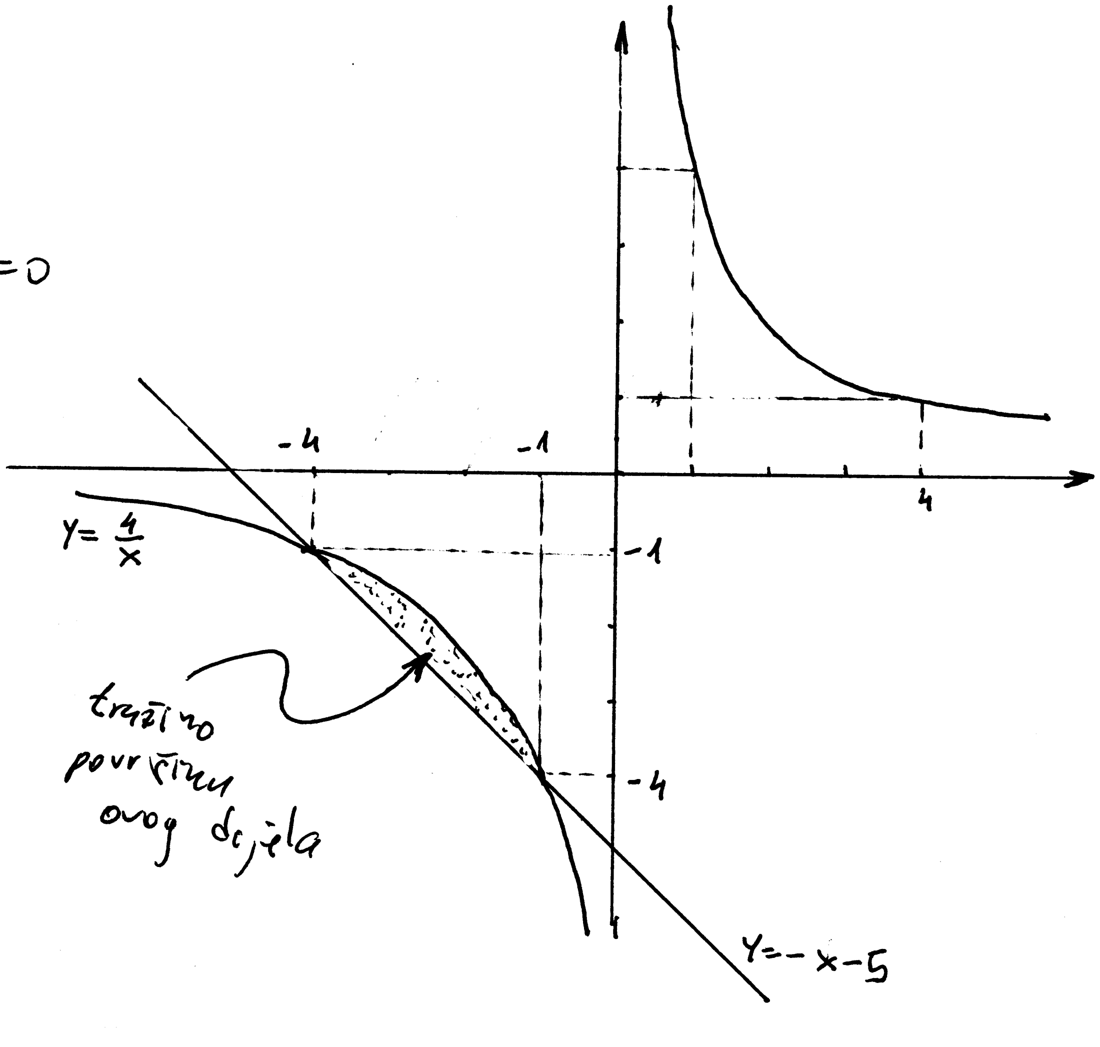
$$x(-x-5)=4$$

$$x^2+5x+4=0$$

$$(x+4)(x+1)=0$$

$x_1=-4 \Rightarrow y_1=-1$

$x_2=-1 \Rightarrow y_2=-4$



tražimo površinu ovog dijela

$$P=P_1-P_2$$

$$P_1=\left|\int_{-4}^{-1}(-x-5)\,dx\right|=\ldots=\left|-\frac{15}{2}\right|=\frac{15}{2}$$

$$P_2=\left|\int_{-4}^{-1}\frac{4}{x}\,dx\right|=\ldots=\left|-8\ln 2\right|=8\ln 2$$

$$P=\frac{15}{2}-8\ln 2$$

tražena površina

(#) Odrediti površinu figure ograničenu parabolom $y=x^2+4x$ i pravom $x-y+4=0$.

Rj.-upute:

Odredimo presječne tačke parabole i prave

$$y=x^2+4x$$
$$y=x+4$$
$$\overline{}$$
$$x^2+4x=x+4$$
$$x^2+3x-4=0$$
$$(x+4)(x-1)=0$$

$x_1=-4 \Rightarrow y_1=0$

$x_2=1 \Rightarrow y_2=5$

$y=x^2+4x=x(x+4)$

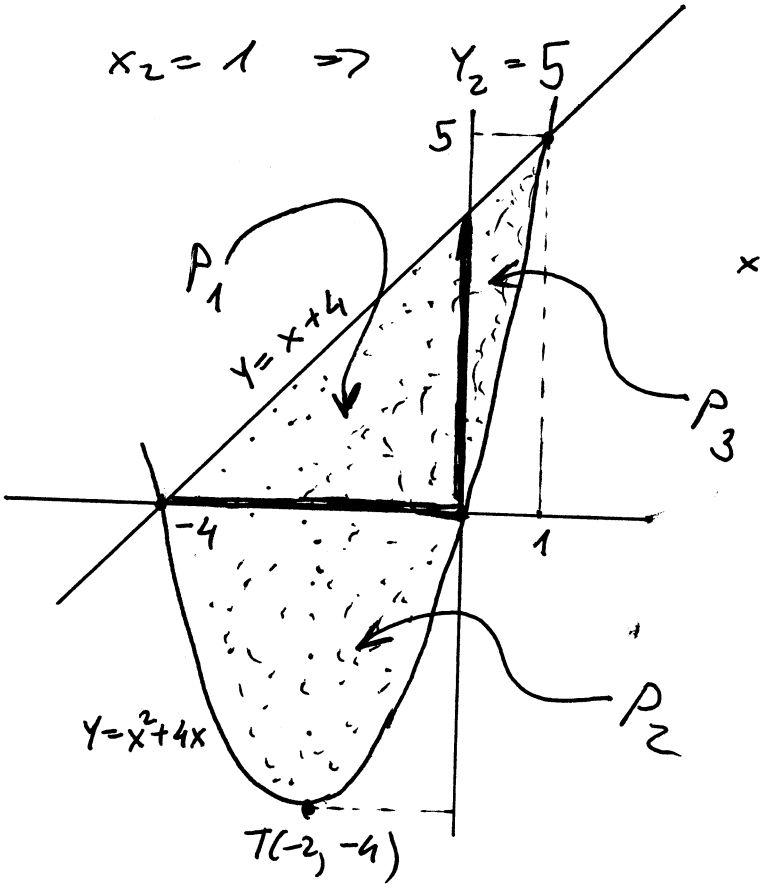
$y'=2x+4$

$2x+4=0$

$x=-2$

$x=-2 \Rightarrow y=-4$

$T(-2;-4)$



$$P=P_1+P_2+P_3$$

$$P_1=\int_{-4}^{0}(x+4)\,dx=\ldots=8$$

$$P_2=\left|\int_{-4}^{0}(x^2+4x)\,dx\right|=\ldots=\left|-\frac{32}{3}\right|=\frac{32}{3}$$

$$P_3=\int_{0}^{1}\left((x+4)-(x^2+4x)\right)dx=\ldots=\frac{13}{6}$$

$$P=P_1+P_2+P_3=8+\frac{32}{3}+\frac{13}{6}=\frac{125}{6}$$ tražena površina

#Odrediti površinu figure ograničenu parabolom $4y=8x-x^2$ i pravom $4y=x+6$.

Rj.-upute:

Prvo odredimo presječne tačke parabole i prave

$$\frac{\begin{array}{l}4y=8x-x^2\\4y=x+6\end{array}}{8x-x^2=x+6}$$

$$x^2-7x+6=0$$

$$D=49-24=25$$

$$x_{1,2}=\frac{7\pm5}{2} \qquad x_1=1 \qquad x_2=6$$

$x_1=1 \Rightarrow 4y=7$

$y_1=\frac{7}{4}$

$x_2=6 \Rightarrow 4y=12$

$y=3$

Presječne tačke prave i parabole su $A(1;\frac{7}{4})$ i $B(6;3)$

$$y=-\frac{1}{4}x^2+2x=x(-\frac{1}{4}x+2)$$

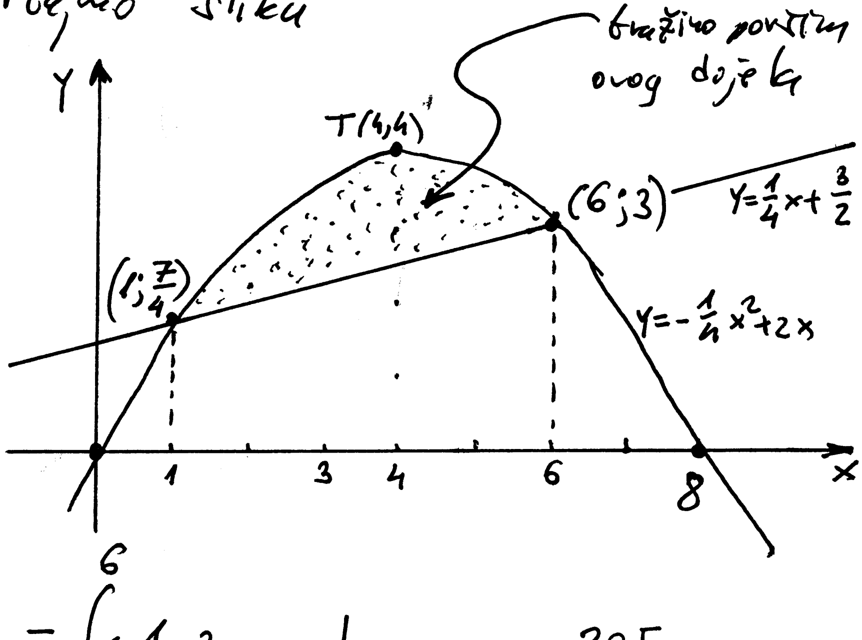
$$y'=-\frac{1}{2}x+2$$

$$-\frac{1}{2}x+2=0 \quad /\cdot 2$$

$$x=4$$

$$T(4,4)$$

Nacrtajmo sliku



$$P=P_1-P_2 \quad \text{gdje je} \quad P_1=\int_1^6\left(-\frac{1}{4}x^2+2x\right)dx=\ldots=\frac{205}{12}$$

$$P_2=\int_1^6\left(\frac{1}{4}x+\frac{3}{2}\right)dx=\ldots=\frac{95}{8}$$

$$P=P_1-P_2=\frac{205}{12}-\frac{95}{8}=\frac{410-285}{24}=\frac{125}{24}=5\frac{5}{24}$$

(#) Odrediti površinu figure ograničene hiperbolom $xy=6$ i pravom $y=7-x$.

Rj.-upute:
Prvo odredimo presječne tačke date hiperbole i prave

$$\frac{\begin{array}{l} xy=6 \\ y=7-x \end{array}}{}$$

$x(7-x)=6$

$7x-x^2=6$

$x^2-7x+6=0$

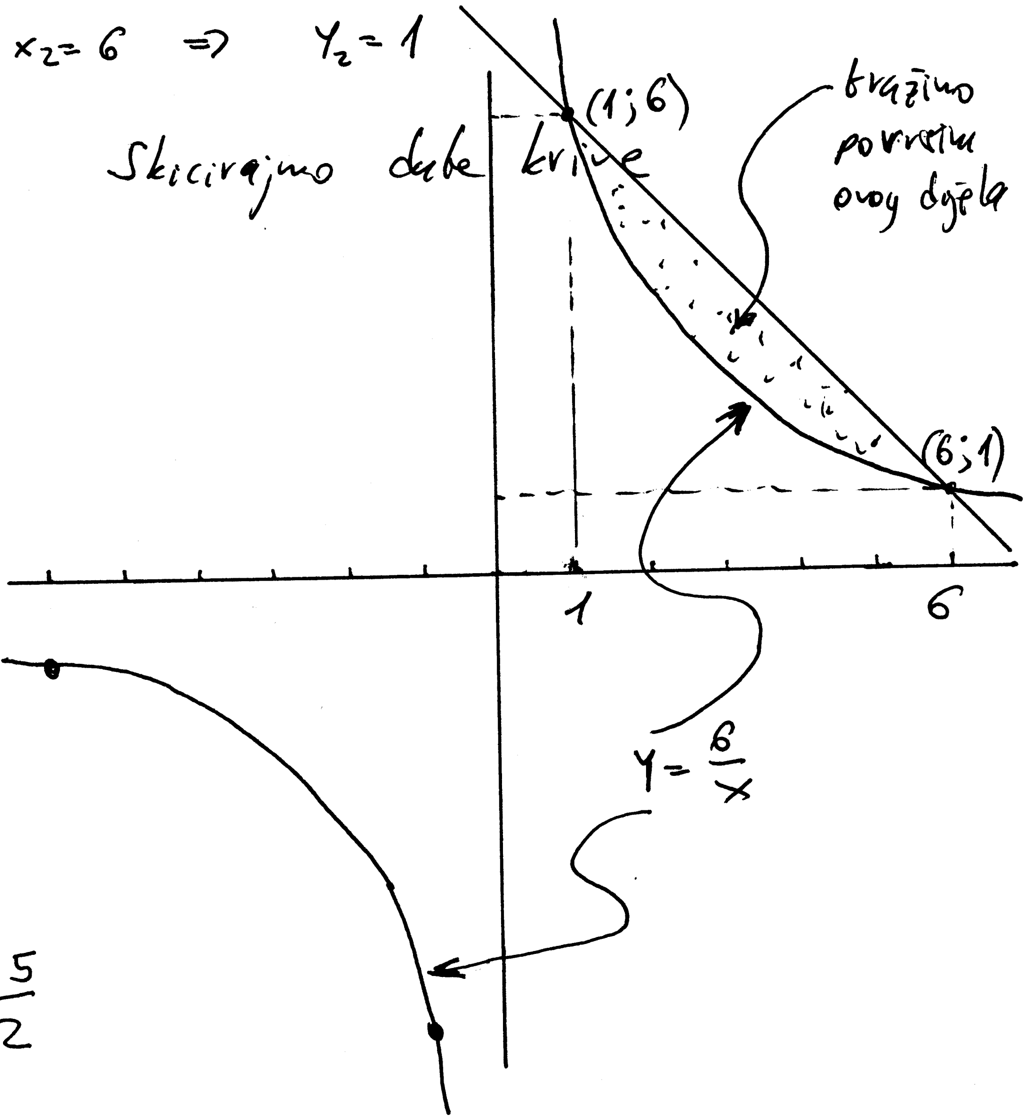
$D=49-24=25$

$x_{1,2}=\frac{7\pm5}{2}$

$x_1=1 \Rightarrow y_1=6$

$x_2=6 \Rightarrow y_2=1$

Skicirajmo date krive



$P=P_1-P_2$

$$P_1=\int_1^6(7-x)dx=\ldots=\frac{35}{2}$$

$$P_2=\int_1^6\frac{6}{x}dx=\ldots=6\ln 6$$

$$P=\frac{35}{2}-6\ln 6$$

(#) Riješiti diferencijalnu jednačinu $y-xy'-\frac{1}{2}y'^2=0$.

Rj.

$$y-xy'-\frac{1}{2}y'^2=0$$

$$y=xy'+\frac{1}{2}y'^2$$

Jednačine oblika $y=xy'+f(y')$ se nazivaju Clairaut-ove dif. jed

uvodimo smjenu $y'=p$

$$dy=p\,dx$$

$$y=xp+\frac{1}{2}p^2 \quad /d$$

$$\underbrace{dy}_{=p\,dx}=\underbrace{p\,dx}+x\,dp+p\,dp$$

$$(x+p)\,dp=0$$

(a) $dp=0$

$$p=C$$

$$y=xC+\frac{1}{2}C^2$$

je opšte rješenje diferencijalne jednačine

(b) $x+p=0$

$$p=-x$$

$$y=xp+\frac{1}{2}p^2$$

$$y=-x^2+\frac{1}{2}x^2$$

$$y=-\frac{1}{2}x^2$$

singularno rješenje diferencijalne jednačine

(#) Riješiti diferencijalnu jednačinu $y'^2 - xy' + y = 0$.

Rj: $y'^2 - xy' + y = 0$

$$y = xy' - y'^2$$

Jednačina oblika $y = xy' + f(y')$ se naziva Clairautova dif.jedn.

uvodimo smjenu $y' = p \Rightarrow dy = p\,dx$

$$y' = \frac{dy}{dx}$$

$$y = xp - p^2 \quad /d$$

$$dy = p\,dx + x\,dp - 2p\,dp$$

$$\underline{\underline{p\,dx}} = \underline{\underline{p\,dx}} + x\,dp - 2p\,dp$$

$$(x - 2p)\,dp = 0$$

(a) $dp = 0$

$p = c \Rightarrow y = xc - c^2$

je opšte rješenje diferencijalne jednačine

(b) $x - 2p = 0$

$$2p = x$$

$$p = \frac{x}{2}$$

$$y = xp - p^2$$

$$y = \frac{1}{2}x^2 - \frac{1}{4}x^2$$

$$y = \frac{1}{4}x^2$$

singularno rješenje diferencij. jedn.

(#) Riješiti diferencijalnu jednačinu $(y-y'x)^2=1+y'^2$.

Rj: $y-y'x=\pm\sqrt{1+y'^2}$

$$y=y'x\pm\sqrt{1+y'^2}$$

Jednačina oblika $y=xy'+f(y')$ se naziva Clairat-ova dif. jedn. i ove diferencijalne jednačine rješavamo na potpuno isti način kao što smo rješavali Lagrange-ove difer. jednač. uvodimo smjenu $y'=p$. ($dy=p\,dx$, $x=uv$).

$y=y'x\pm\sqrt{1+y'^2}$ ovo je Clairaut-ova dif. jedn.

$y'=p$, $y'=\frac{dy}{dx} \Rightarrow dy=p\,dx$

$$y=px\pm\sqrt{1+p^2} \quad /d$$

$$\underset{=p\,dx}{\underline{dy}}=\underline{p\,dx}+x\,dp\pm\frac{2p}{2\sqrt{1+p^2}}dp$$

$$\left(x\pm\frac{p}{\sqrt{1+p^2}}\right)dp=0$$

(a) $dp=0$

$$p=c$$

$$(y-cx)^2=1+c^2$$

opšte rješenje diferencijalne jedn.

(b) $x\pm\frac{p}{\sqrt{1+p^2}}=0$

$$\pm\frac{p}{\sqrt{1+p^2}}=-x \quad /^2$$

$$\frac{p^2}{1+p^2}=x^2$$

$$p^2=\underbrace{x^2(1+p^2)}_{x^2+x^2p^2}$$

$$(1-x^2)p^2=x^2$$

$$p^2=\frac{x^2}{1-x^2}$$

$$p=\frac{x}{\sqrt{1-x^2}}$$

$$y=\frac{x^2}{\sqrt{1-x^2}}\pm\sqrt{1+\frac{x^2}{1-x^2}}=\frac{x^2\pm1}{\sqrt{1-x^2}}$$

singularno rješenje diferenc. jednačine

(#) Riješiti diferencijalnu jednačinu $y = y'x + \sqrt{4+y'^2}$.

Rj. Jednačina oblika $y = xy' + f(y')$ se naziva Clairaut-ova diferencijalna jednačina i ove diferencijalne jednačine rješavamo na potpuno isti način kao što smo rješavali Lagrange-ove diferencijalne jednačine – uvodimo smjenu

$$y' = p, \quad \left[y' = \frac{dy}{dx} \right.$$

$$dy = p\,dx$$

$$x = uv$$

$$y = y'x + \sqrt{4+y'^2}$$ Clair. dif. jedn.

$$y' = p \Rightarrow dy = p\,dx$$

$$y = px + \sqrt{4+p^2} \quad /d$$

$$dy = p\,dx + x\,dp + \frac{2p\,dp}{2\sqrt{4+p^2}}$$

$$\underline{p\,dx} = \underline{p\,dx} + x\,dp + \frac{p\,dp}{\sqrt{4+p^2}}$$

$$\left(x + \frac{p}{\sqrt{4+p^2}} \right) dp = 0$$

(a) $dp = 0 \Rightarrow p = c$

$$y = cx + \sqrt{4+c^2}$$

opšte rješenje diferenc. jednačine

(b) $x + \frac{p}{\sqrt{4+p^2}} = 0$

$$\frac{p}{\sqrt{4+p^2}} = -x \quad /^2$$

$$\frac{p^2}{4+p^2} = x^2$$

$$p^2 = x^2(4+p^2)$$

$$p^2 = x^2p^2 + 4x^2$$

$$(1-x^2)p^2 = 4x^2$$

$$p^2 = \frac{4x^2}{1-x^2}$$

$$p = \frac{2x}{\sqrt{1-x^2}}$$

$$y = px + \sqrt{4+p^2} = \frac{2x^2}{\sqrt{1-x^2}} + \sqrt{4 + \frac{4x^2}{1-x^2}} = \frac{2x^2}{\sqrt{1-x^2}} + \sqrt{\frac{4-4x^2+4x^2}{1-x^2}}$$

$$y = \frac{2x^2+2}{\sqrt{1-x^2}}$$

singularno
rješenje diferencijalne
jednačine (rješenje koje se ne može dobiti iz općeg rješenj